Optimal Lossless Source Codes for Timely Updates

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Joint work with

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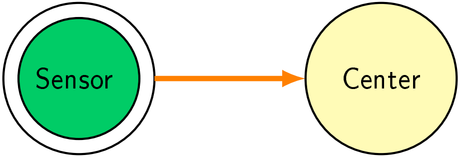
Indian Institute of Science

Motivation



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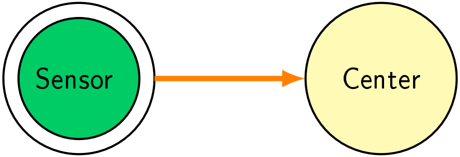
Source - The Hindu



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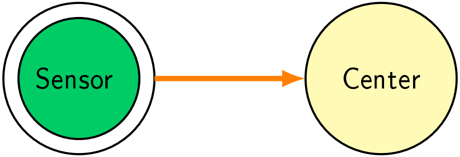
Timely Updates are critical.



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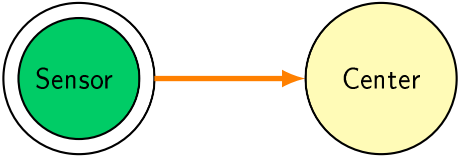

Timely Updates are critical.

Age of Information[1] - metric to capture timeliness.

[1] Kaul, S., Yates, R., and Gruteser, M. (2011, December). On piggybacking in vehicular networks. In Global Telecommunications Conference (GLOBECOM 2011), 2011 IEEE (pp. 1-5). IEEE.

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- AOI: Time lag between the latest information at the RX w.r.t. that at TX.

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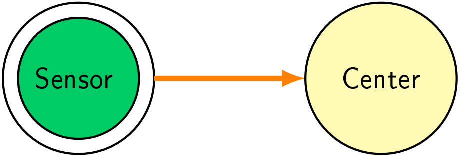

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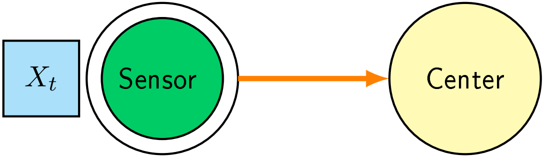

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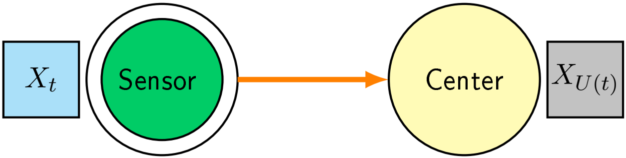

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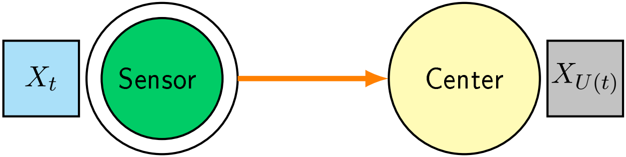

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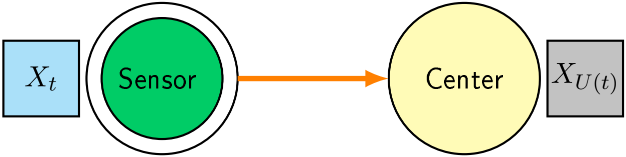

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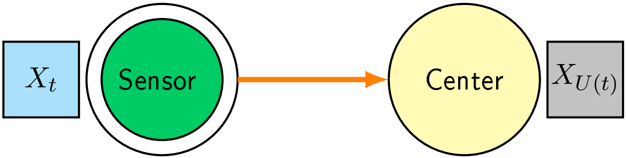

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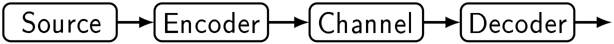
- We restrict to Memoryless Update Schemes.

Memoryless Update Schemes



iid P Prefix-free Noiseless
1 Bit/Time slot

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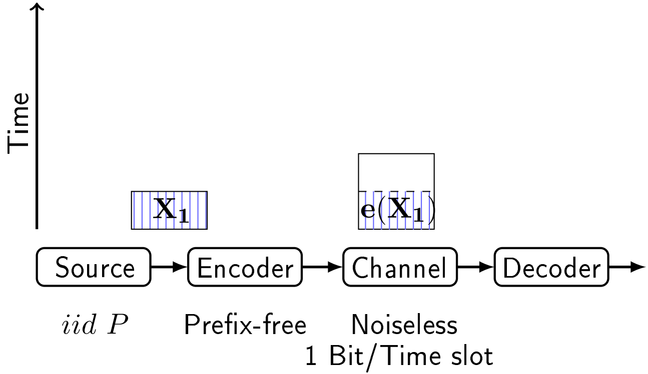

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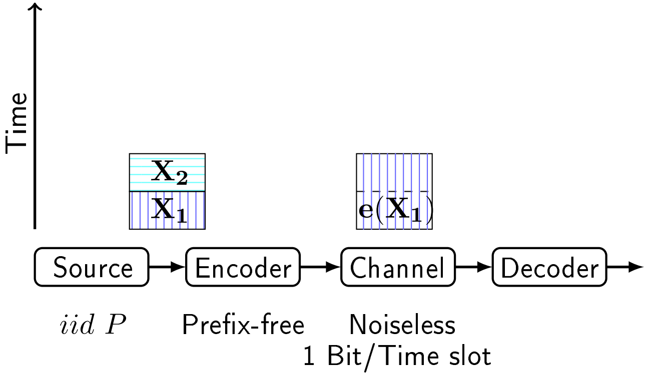

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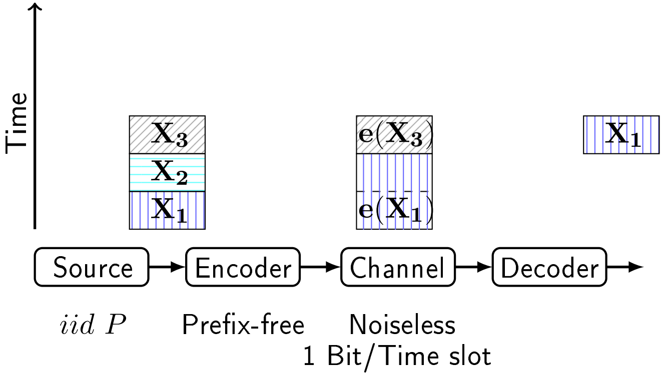

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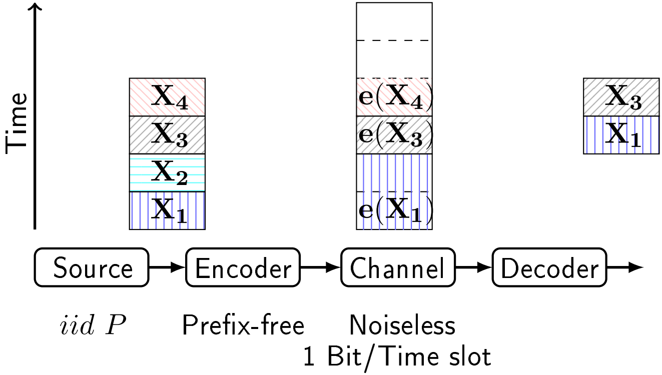

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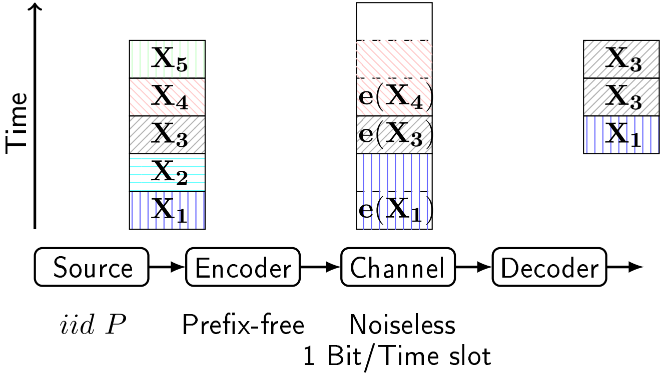

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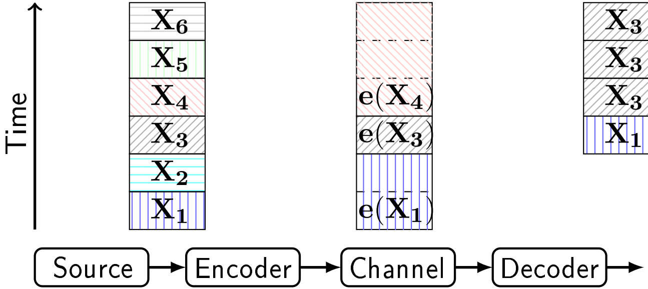

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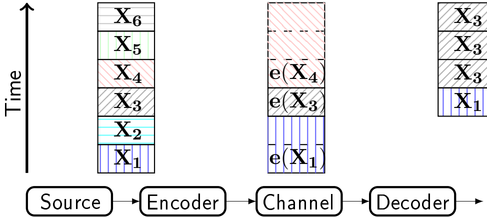
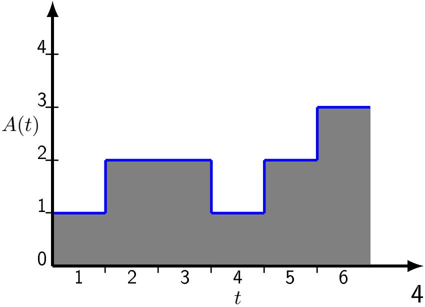
Illustration of Instantaneous Age



$$A(t) = t - U(t)$$

$U(t) =$ Index of latest information at the decoder



Characterization of Average Age

$$\bar{A}(e) \triangleq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} A(t)$$

$\ell(x) \triangleq$ code-length for a symbol x, $L \triangleq \ell(X)$.

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For a prefix-free code e, $\bar{A}(e) = \mathbb{E}\left[L\right] + \frac{\mathbb{E}\left[L^2\right]}{2\mathbb{E}[L]} - \frac{1}{2} \quad a.s.$.

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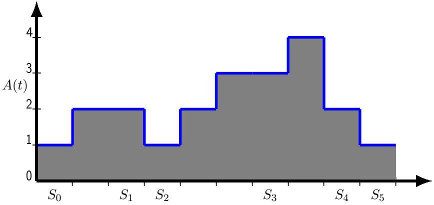
Proof Idea:

- $S_i \triangleq i^{th}$ reception



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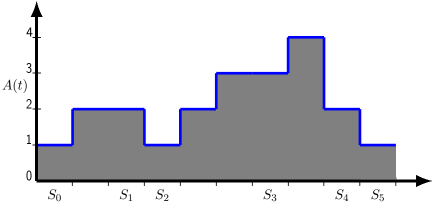

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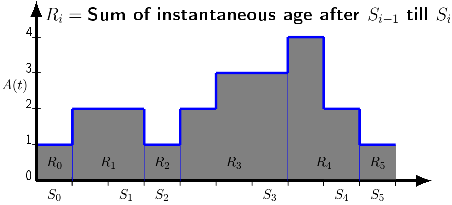

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- $(R_{2i+1})_{i\in\mathbb{N}}$ is iid,
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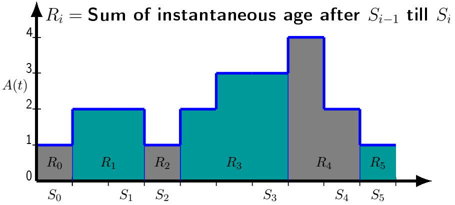

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Are Shannon Codes Optimal?

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Example: Consider $\mathcal{X} = \{0, ..., 2^n\}$ and a pmf P on $\mathcal{X}$ given by

$$P(x) = \begin{cases} 1 - \frac{1}{n}, & x = 0 \\ \frac{1}{n2^n}, & x \in \{1, \ldots, 2^n\}. \end{cases}$$

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Shannon codes are order-wise suboptimal!

Our Approach

Reduction to a simpler problem

Need to solve IP;

$$\min \mathbb{E}[L] + \frac{\mathbb{E}[L^2]}{2\mathbb{E}[L]}$$

$$\text{s.t.} \quad \ell \in \mathbb{Z}_+^{|\mathcal{X}|},$$

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Proposition

Cost using this approach will be atmost 2.5 bits away from the optimal cost.

Structural Result for RP

Real valued Shannon lengths for P: $\ell(x) = -\log P(x) \quad \forall x$

Main Theorem

Optimal solution for RP is unique and is given by

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where P^* is a tilting of source distribution P.

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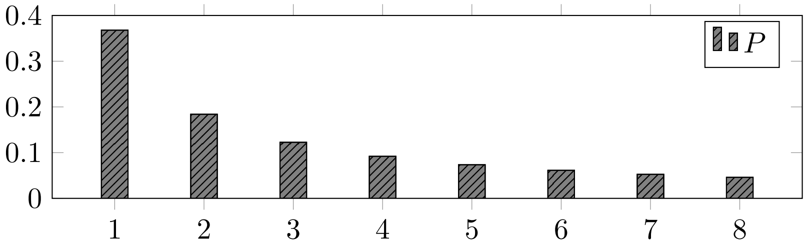

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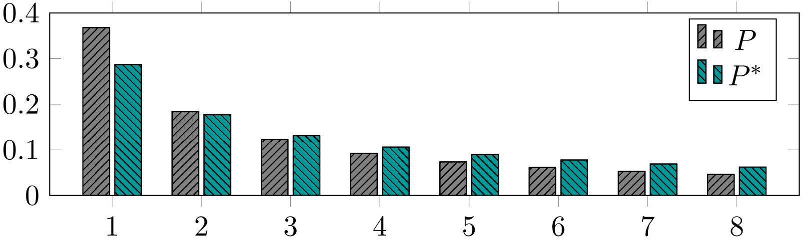

Proof sketch of Main theorem

- Main step: Linearizing the Average Age Cost

$$\mathbb{E}\left[L\right] + \frac{\mathbb{E}\left[L^2\right]}{2\mathbb{E}\left[L\right]} = \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} g(y, x)\ell(x)$$

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$$\Delta^* = \min_{\ell \in \Lambda} \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} g(y, x)\ell(x)$$

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2. Inner $\min$ is attained by $\ell'(x) = -\log P'(x)$ for $P'(x) \propto g(y,x)$
3. Use Entropy Maximization to find the least-favorable y

$$\Delta^* = \max_{\substack{y \in \mathcal{Y}, \\ g(y,\cdot) \geq 0}} \sum_{x \in \mathcal{X}} g(y,x) \log \frac{\sum_{x \in \mathcal{X}} g(y,x)}{g(y,x)}$$

Proof sketch of Main theorem

- Main step: Linearizing the Average Age Cost

$$\mathbb{E}[L] + \frac{\mathbb{E}[L^2]}{2\mathbb{E}[L]} = \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} g(y,x)\ell(x)$$

- The wrap-up

1. Minimax claim

$$\Delta^* = \min_{\ell \in \Lambda} \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} g(y,x)\ell(x) = \max_{\substack{y \in \mathcal{Y}, \\ g(y,\cdot) \geq 0}} \min_{\ell \in \Lambda} \sum_{x \in \mathcal{X}} g(y,x)\ell(x)$$

2. Inner $\min$ is attained by $\ell'(x) = -\log P'(x)$ for $P'(x) \propto g(y,x)$

3. Use Entropy Maximization to find the least-favorable y

$$\Delta^* = \max_{\substack{y \in \mathcal{Y}, \\ g(y,\cdot) \geq 0}} \sum_{x \in \mathcal{X}} g(y,x) \log \frac{\sum_{x \in \mathcal{X}} g(y,x)}{g(y,x)}$$

Minimizing lengths for the least-favorable y are optimal

Main Step: Linearizing the Average Age Cost

- Linearizing the rational form (easy):

$$\mathbb{E}\left[L\right] + \frac{\mathbb{E}\left[L^2\right]}{2\mathbb{E}\left[L\right]} = \max_{z \geq 0} \left(1 - \frac{z^2}{2}\right) \mathbb{E}\left[L\right] + z\sqrt{\mathbb{E}\left[L^2\right]}$$

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- Linearizing the 2-norm term?

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- Linearizing the 2-norm term?

A new variational formula for p-norm of a random variable

$$\|X\|_p = \max_{Q \ll P} \mathbb{E}\left[\left(\frac{dQ}{dP}\right)^{\frac{p-1}{p}} |X|\right]$$

Main Step: Linearizing the Average Age Cost

- Linearizing the rational form (easy):

$$\mathbb{E}[L] + \frac{\mathbb{E}[L^2]}{2\mathbb{E}[L]} = \max_{z \geq 0} \left(1 - \frac{z^2}{2}\right)\mathbb{E}[L] + z\sqrt{\mathbb{E}[L^2]}$$

- Linearizing the 2-norm term?

A new variational formula for 2-norm of a random variable

$$\sqrt{\mathbb{E}[L^2]} = \max_{Q \ll P} \sum_{x \in \mathcal{X}} \sqrt{Q(x)P(x)}\ell(x)$$

Main Step: Linearizing the Average Age Cost

- Linearizing the rational form (easy):

$$\mathbb{E}\left[L\right] + \frac{\mathbb{E}\left[L^2\right]}{2\mathbb{E}\left[L\right]} = \max_{z\geq 0}\left(1 - \frac{z^2}{2}\right)\mathbb{E}\left[L\right] + z\sqrt{\mathbb{E}\left[L^2\right]}$$

- Linearizing the 2-norm term?

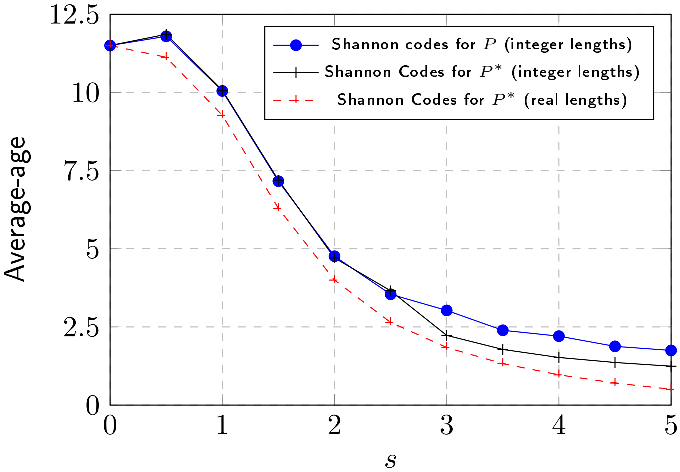
A new variational formula for 2-norm of a random variable

$$\sqrt{\mathbb{E}\left[L^2\right]} = \max_{Q\ll P}\sum_{x\in\mathcal{X}}\sqrt{Q(x)P(x)}\ell(x)$$

$$\mathbb{E}\left[L\right] + \frac{\mathbb{E}\left[L^2\right]}{2\mathbb{E}\left[L\right]} = \max_{z\geq 0,\, Q\ll P}\left(1 - \frac{z^2}{2}\right)\mathbb{E}\left[L\right] + z\sum_{x\in\mathcal{X}}\sqrt{Q(x)P(x)}\ell(x)$$

Simulation Results

$\texttt{Zipf}(s, N)$ is given by $P(i) = \frac{i^{-s}}{\sum_{j=1}^{N} j^{-s}}, \quad 1 \leq i \leq N.$



Comparison of proposed codes and Shannon codes for $\texttt{Zipf}(s, 256)$ w.r.t. s.

A related problem

- How to design source-codes for Minimum Queuing Delay?[2]

[2] Humblet, P. A. (1978). Source coding for communication concentrators.

A related problem

- How to design source-codes for Minimum Queuing Delay?[2]
- Cost Function: $\bar{D}(e) = \begin{cases} \mathbb{E}[L] + \frac{\lambda \mathbb{E}[L^2]}{2(1-\lambda \mathbb{E}[L])}, & \lambda \mathbb{E}[L] < 1, \\ \infty, & \lambda \mathbb{E}[L] \geq 1. \end{cases}$

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  Codes which minimize the first moment are "robust".

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- We formally prove this empirical observation using our recipe.

Structural solution for the relaxed problem

$\ell^*(x) = -\log P^*(x)$, where P^* satisfies

$$D(P||P^*) \leq \log\left(1 + \frac{1}{\sqrt{2}}\right).$$

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In summary ...

- New variational formula for p^{th} norm of random variable
- Recipe for minimizing average age based on Entropy Maximization
- General Recipe: can be used to optimize other non-linear costs

Backup Slides

Similar Cost Function

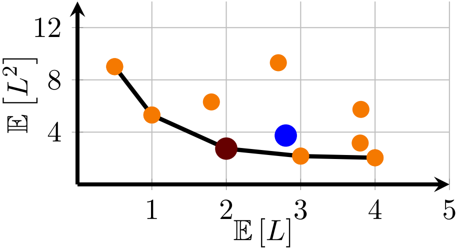
Minimum Delay Problem[3]

$$\bar{D}(e) = \begin{cases} \mathbb{E}[L] + \frac{\lambda \mathbb{E}[L^2]}{2(1-\lambda \mathbb{E}[L])}, & \lambda \mathbb{E}[L] < 1, \\ \infty, & \lambda \mathbb{E}[L] \geq 1. \end{cases}$$

Minimum Age Problem

$$\bar{A}(e) = \mathbb{E}[L] + \frac{\mathbb{E}[L^2]}{2\mathbb{E}[L]} - \frac{1}{2}$$

Convex Hull Algorithm[4]



[3] Humblet, P. A. (1978). Source coding for communication concentrators.

[4] Larmore, L. L. (1989). Minimum delay codes. SIAM Journal on Computing, 18(1), 82-94.

Performance of Shannon Codes

Shannon code for P: $\ell(x) = \lceil -\log P(x) \rceil \quad \forall x$.

Lemma

Given a pmf P on $\mathcal{X}$, a Shannon code e for P has average age at most $O(\log |\mathcal{X}|)$.